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What Do We Know about Self-Similarity in Fluid Turbulence?

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The evidence is reviewed on the statistical behavior of the small-scale fluctuations in high-Reynolds-number fluid turbulence. The qualitative phenomenological information is summarized and the predictions of the 1941 Kolmogorov theory are reviewed. Then direct numerical simulation and its role in suggesting dynamical mechanisms are briefly discussed. Finally, the evidence on the multifractal structure of the dissipation field is reviewed. It is concluded that the experimental evidence for some kind of dynamical self-similarity is strong, but that there has been essentially no progress in fundamental theoretical understanding of the underlying mechanisms.

KEY WORDS: Navier–Stokes equations; turbulent cascades; scaling exponents; multifractal structure.

Big whorls have little whorls
which feed on their velocity.
Little whorls have lesser whorls
and so on to viscosity.
(in the molecular sense)

L. F. Richardson, 1922

1. INTRODUCTION

Richardson's famous ditty, or the collected works of G. I. Taylor from as early as 1915, give an essentially correct qualitative picture of the dynamics of strongly turbulent flow. Kolmogorov's 1941 theory[1] added some dynamical content to this picture. Its basic assumption is that ε, the average rate of energy dissipation per unit mass, is independent of viscosity in the limit of zero viscosity. This idea is central to all engineering modeling

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of turbulence, since it allows the large-scale properties of the flow to be computed approximately without a detailed understanding of the small-scale turbulent fluctuations. This same idea also has deep mathematical implications about the zero-viscosity limit of the Navier–Stokes equations. In this paper I review what we know firmly, which is very little. I then review the phenomenological evidence for some kind of self-similarity in turbulence. I briefly summarize some clues from direct numerical simulation, and very briefly discuss the role of statistical theories. I conclude by suggesting that dynamical self-similarity and some kind of cascade mechanism are probably contained within the Navier–Stokes equations, but that fundamental theory has so far made little progress in support of this conclusion.

2. WHAT IS THE PROBLEM?

If thermal effects can be neglected, and pressure-induced density changes are small, a fluid flow can be considered incompressible. Consider, for example, the flow past a cylinder. The free stream velocty far upstream is U, and the diameter of the cylinder is L. The fluid is assumed to be Newtonian, which is an excellent approximation for ordinary fluids such as air or water. The only relevant molecular parameter is the kinematic viscosity ν, which serves as a diffusion coefficient for transverse momentum. For a given flow geometry, the flow is completely characterized by a single dimensionless parameter,

$$\mathrm{Re} = UL/\nu \tag{1}$$

known as the Reynolds number. This parameter is the inverse of an appropriate dimensionless viscosity. When the Reynolds number is large, the flow is turbulent. The wake behind the cylinder exhibits a flow field which is chaotic in both space and time, but whose averages and statistical properties are stable. It is these averages and statistical properties that we want to understand. The problem can be posed in a similar way for a high-Reynolds-number jet or for the turbulent boundary layer over a flat surface. An initial qualitative understanding of turbulent flow fields is best obtained from flow visualizations. An excellent source is the book *An Album of Fluid Motion* assembled by Van Dyke.[2] The reader is encouraged to browse through Chapter 6 on turbulence.

3. WHAT DO WE KNOW FOR SURE?

In a gas, the relevant molecular length scale is the collision mean free path. In a liquid it is the mean distance between molecules. At the molecular level, gases and liquids have little in common. Often, however, fluid motions occur only on length scales large compared to molecular scales, and changes in density are also negligible. In this situation gases and liquids share a common equation of motion, the Navier–Stokes equation of hydrodynamics,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} \tag{2}$$

where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field, ρ is the constant density, $p(\mathbf{r}, t)$ is the dynamic pressure field, and ν is the kinematic viscosity of the fluid. For constant density, local mass conservation gives the condition

$$\nabla \cdot \mathbf{v} = 0 \tag{3}$$

which is statement that the flow (not the fluid) is incompressible. These equations must be supplemented by an appropriate boundary condition. For both gases and liquids this is the no-slip boundary condition that the relative velocity of fluid and solid vanishes at a solid surface. We take this boundary condition as phenomenologically given, recognizing that a full molecular understanding of its origin is far from trivial.[3]

We cannot calculate the details of a turbulent flow directly from the Navier–Stokes equations, but we can make two predictions. First we expect the fluid to be a dissipative dynamical system with a very large number of effective degrees of freedom. Thus, we expect chaotic behavior of some kind as a natural consequence of the dynamics. Second, the statistical properties of the velocity field should be the same if the Reynolds number is the same and the flow geometry is the same. We can rescale velocity U and length L so as to keep Re constant, or we can change from air to water with an appropriate change in UL so as to keep UL/ν constant.

There is considerable theoretical and experimental support for Reynolds number scaling in strong fluid turbulence, and for believing that the Navier–Stokes equations are an adequate starting point for understanding strongly turbulent flows. In gases, kinetic theory tells us that the Navier–Stokes equations should apply to motions on a scale large compared to a collision mean free path. The smallest turbulent eddies, according to the 1941 Kolmogorov theory discussed below, should be no smaller than about 1 mm in any laboratory or geophysical flows in air. This is in good agreement with experiment, and 1 mm is much larger than a typical mean free path in air. In water the Navier–Stokes equations should apply if the scales of motion are much larger than a typical molecular diameter. The smallest turbulent eddies observed are no smaller than about 0.1 mm, which is again much larger than a molecular scale.

Perhaps most important is that the observed statistical properties of turbulence, for a given flow geometry and a given Reynolds number, are the same in air as in water. These two fluids are very different at a molecular level. They share very little except the Navier-Stokes equations.

4. WHAT DO WE KNOW APROXIMATELY?

Certain general properties are observed to be common to all high-Reynolds-number turbulent flows. For convenience we consider only those flows which are statistically steady. The time-averaged velocity and the mean square velocity fluctuations are nearly independent of Reynolds number for large Reynolds number. The range of turbulent "eddy sizes" r excited (to be defined more precisely later) is very broad, extending from a largest scale of the order of the external length scale L, to a smallest scale η which must be determined from a dynamical theory. The observed ratio of these scales is well approximated by

$$L/\eta = \text{const} \cdot (\text{Re})^{3/4} \tag{4}$$

Equation (4) is a well-known consequence of the 1941 Kolmogorov theory, but for now we take it as an observed property. For an atmospheric flow with $\text{Re} = 10^6$, this implies a ratio of largest to smallest scales of about 10^4.

The small scales of the flow, $r \ll L$, exhibit some degree of universality. They exhibit approximate statistical isotropy, and their statistical properties, when appropriately scaled, are independent of the large-scale flow geometry.

I emphasize that there are two distinct kinds of universality. The large-scale properties of the flow depend on geometry, but are independent of Reynolds number. The small-scale fluctuations depend on Reynolds number, but are, in a sense I will now make more precise, independent of flow geometry. Both of these properties suggest a certain degree of insulation of the large-scale flow properties from the small-scale fluctuations.

5. THE 1941 KOLMOGOROV THEORY

To make the preceding picture quantitative, assume that energy cascades from large scales to small scales, that viscosity is important only at the smallest scales, and that the dynamics of the cascade is governed entirely by the average rate at which energy is being transferred. The average rate of energy dissipation per unit mass ε plays several roles. It is the rate at which energy is fed into the turbulence at large scales from the mean flow. It is the rate at which energy is transferred from large to small scales by the nonlinear terms in the Navier–Stokes equations, and it is also the rate at which energy is dissipated due to viscosity at small scales.

Denoting the position in the fluid by $\mathbf{r} = (x, y, z)$ and the velocity vector by $\mathbf{v} = (u, v, w)$, and assuming statistical isotropy at the scales where dissipation occurs, one finds that ε is given by

$$\varepsilon = 15\nu \left\langle \left(\frac{\partial u}{\partial x}\right)^2 \right\rangle \tag{5}$$

The derivation of Eq. (5) is given in ref. 1. The essential features are first to write the mean energy dissipation in an incompressible fluid in terms of the mean square rate of strain. Then relations for isotropic turbulence such as

$$\langle (\partial u/\partial y)^2 \rangle = 2\langle (\partial u/\partial x)^2 \rangle$$

are used, and the relevant terms are collected.

In Eq. (5) $\langle \cdot \rangle$ denotes a time average over a statistically steady flow. Equation (5) gives the dissipation rate ε in terms of directly measurable quantities. In terms of large-scale quantities the same quantity is given by

$$\varepsilon = \text{const} \cdot (U^3/L) \tag{6}$$

where the constant depends on the geometry of the flow, and cannot be calculated without a full theory of turbulence. I make the essential assumption, however, that the dissipation rate is independent of viscosity in the limit of small viscosity. The viscosity serves only as a sink for the turbulent energy. The scales at which dissipation occur adjust to the value of the viscosity, but the amount of energy dissipated is constant.

The length scale η at which dissipation occurs can be calculated from dimensional analysis. The only length which can be formed from the kinematic viscosity ν and the energy dissipation rate ε is

$$\eta = (\nu^3/\varepsilon)^{1/4} \tag{7}$$

This quantity is known as the Kolmogorov microscale, and Eq. (7) is a simple and experimentally verifiable prediction. It is equivalent to Eq. (4), and it works very well in practice.[4]

For high enough Reynolds numbers, there will be a range of length scales r satisfying

$$\eta \ll r \ll L \tag{8}$$

This is known as the inertial subrange. Since $r \ll L$, I assume universal isotropic behavior independent of the flow geometry. Since $r \gg \eta$, I assume that viscosity plays no role. I want to calculate the mean kinetic energy in an "eddy" of size r, defined by

$$C(r) = \langle [u(x+r) - u(x)]^2 \rangle \tag{9}$$

The only quantity with the dimensions of velocity which one can form using only ε and r is $(\varepsilon r)^{1/3}$, so that $C(r)$ is given by

$$C(r) = \text{const} \cdot (\varepsilon r)^{2/3} \tag{10}$$

For sufficiently small r, when dissipation becomes important, the flow field becomes smooth, and one can evaluate Eq. (9) keeping only the first term in a Taylor series expansion. Using Eq. (5), one thus obtains

$$C(r) = \varepsilon r^2/15\nu \tag{11}$$

Equation (11) applies in principle only to the limited range where r is small compared to the dissipation scale η, but still large compared to molecular scales. In practice, the crossover from Eq. (10) to Eq. (11) occurs at distances somewhat larger than η.

Equations (10) and (11) are most conveniently expressed in terms of a scaling form for the energy spectrum $E_1(k)$ defined by

$$\langle u(x)\, u(x+r) \rangle = \int_0^\infty E_1(k) \cos(kr)\, dk \tag{12}$$

In particular, the total kinetic energy and the total rate of energy dissipation are given, respectively, by

$$\langle u^2 \rangle = \int_0^\infty E_1(k)\, dk, \qquad \varepsilon = 15\nu \int_0^\infty E_1(k)\, k^2\, dk \tag{13}$$

An essential feature of high-Reynolds-number turbulence is that the kinetic energy is concentrated at large scales ($k \sim 1/L$), and that the dissipation is concentrated at small scales ($k \sim 1/\eta$).

The predictions of the Kolmogorov theory are conveniently summarized in the scaling law

$$E_1(k) = \varepsilon^{2/3} k^{-5/3} f(k\eta) \tag{14}$$

where the scaling function $f(x)$ is constant for small x, and decays rapidly for large x. A quantitative dynamical theory is required to obtain the scaling function.

We have obtained Eqs. (7) and (10) by simple dimensional analysis. An equivalent argument can be expressed in terms of the relevant time scales, and gives some elementary dynamical understanding. Let $\Delta u(r) = [C(r)]^{1/2}$ be a typical velocity for an eddy of size r. Let $\tau(r)$ be a characteristic time for energy transfer from eddies of size r to smaller scales. The average rate of energy transfer ε is given by $[\Delta u(r)]^2/\tau(r)$, and is independent of r when viscosity can be neglected. If we assume that the characteristic time $\tau(r)$ is given by the eddy turnover time $r/\Delta u(r)$ at scale r, we obtain $\varepsilon = [\Delta u(r)]^3/r$, which is equivalent to Eq. (10). Viscosity will become important when the viscous diffusion time r^2/ν becomes comparable to the eddy turnover time $r/\Delta u(r)$. Using Eq. (10), this leads to Eq. (7).

Equation (14) tells us nothing about the large-scale properties of the flow, which are nonuniversal, anisotropic, and of dominant practical importance. Even for the small scales, it is incomplete since it deals only at the level of the simple correlation function $\langle u(x)\, u(x+r)\rangle$. Despite this, it is successful in collapsing a large amount of data from a wide variety of flows to a single apparently universal curve. This is summarized in ref. 4. In the inertial range, the scaling function $f(x) = f(0)$. The constant $f(0)$ is approximately equal to 0.5, and is known experimentally to about 10 %. For large values of x, the scaling function $f(x)$ decreases in an approximately exponential manner, and is again universal, with variations of the order of 10–20 % among widely varying flows. Statistical theories of turbulence have had considerable success in calculating this scaling function, but these theories are not discussed in this paper.

6. DIRECT NUMERICAL SIMULATION

The direct numerical simulation of high-Reynolds-number turbulence is unfortunately not feasible. If all scales down to the dissipation scale are to be resolved, the number of coupled differential equations which must be solved is of the order of $(L/\eta)^3$ in three dimensions. Using the Kolmogorov theory, this scales as the 9/4 power of the Reynolds number. To simulate homogeneous turbulence for a few eddy turnover times in a few hours of supercomputer time, one can handle only about 10^6 differential equations. This limits calculations to modest Reynolds numbers. It is not feasible to expect direct computation of a cascade with a wide range of length scales.

Yet direct simulation can give important insights into the dynamics of turbulence. If one starts with a few randomly oriented large eddies in a box with periodic boundary conditions, one has a reasonable starting point for homogeneous turbulence. For a time of the order of a large eddy turnover time, very little dissipation is seen. Then small-scale activity is generated, and the rate of energy dissipation rises to a plateau. Since the turbulence is not externally forced, the total kinetic energy and the dissipation rate then slowly decrease, but maintain a quasi-steady state. Even the first hint of a "five-thirds law" is seen, but the Reynolds numbers are sufficiently small that no well-defined inertial range is expected. Perhaps most important, direct simulation can help one understand the dynamical mechanisms by which small scales are formed, and the possible appearance of spatial singularities in the solutions.

A qualitative feature of the Kolmogorov theory is that the characteristic time $\tau(r)$ decreases with decreasing scale size. We can estimate the total time for energy to cascade to infinite wave number in the absence of viscosity. Suppose that each cascade step is a factor of two in size from $r_n = 2^{-n}L$ to $r_{(n+1)} = r_n/2$. Using Eq. (10) to estimate the eddy turnover time tells us that the characteristic time scale at the nth cascade step is $2^{-2n/3}$ times the large eddy turnover time (L/U). The total time to reach zero scale size is then some multiple of (L/U) obtained by summing the simple geometric series.

The preceding argument suggests that the inviscid limit of the Navier–Stokes equations should develop spatial singularities in a finite length of time. Whether this is in fact the case remains an open question, both theoretically and computationally. Although the connection between singularities of the Euler equations and real turbulent flows is a loose one, the existence of singularities is of considerable fundamental interest. An interesting model problem, for example, is the singular behavior of two antiparallel vortex filaments as they approach each other. This has been carefully studied by Pumir and Siggia.[5] More recent computations by Pumir and Kerr[6] strongly suggest, however, that severe flattening of these filaments occurs before any singular region is reached. This suggests that the important dynamics of small scales in turbulence may be dominated by sheets or ribbons of vorticity rather than vortex tubes. This could be important in dynamically understanding how vorticity is stretched to give the small-scale structures of high-Reynolds-number turbulence.

Kerr[7] has carried out a detailed statistical analysis of the vortex structures that occur in modest-Reynolds-number homogeneous turbulence. From pointwise statistics, he infers characteristic pancake structures with one compressing direction and two stretching directions. The ratio of principal values of the rate of strain tensor in these structures is typically $3:1:-4$, adding to zero as it must for incompressible flow. The vorticity is found to align preferentially along the intermediate positive rate of strain.

If the structures seen by Kerr are truly characteristic of small-scale turbulence, many interesting physical questions are raised. What is the dynamical mechanism by which the Navier–Stokes equations lead to these structures? What happens at larger Reynolds numbers? Do these structures fold over, break up, or in any way act as the initial step of a cascade process? Is there any natural mechanism by which dynamical self-similarity arises? Does one have to invoke a sequence of instabilities or is there some sort of smooth dynamical process leading to smaller and smaller scales?

In any case the initial formation of small scales at modest Reynolds numbers is far from the simple picture suggested by Richardson's ditty. If sheetlike structures are formed, the picture of compact objects breaking up into smaller compact objects is qualitatively wrong. There is something more like a stretching and folding of ribbons, but so far we have no dynamical understanding of how this happens. Although I have used recent numerical work to pose this problem, I could have used earlier work. An excellent earlier reference to the essential physical problem is Kraichnan's[8] 1974 paper. I return now to high-Reynolds-number experiments, with the objective of understanding the geometry of the small scales in turbulence.

7. THE MULTIFRACTAL STRUCTURE OF TURBULENCE

The spectral content of high-Reynolds-number turbulence is well summarized by Eq. (14), but a direct examination of a velocity derivative signal suggests a very complicated structure for higher order correlation functions. Typically, $u(t)$, the component of the velocity along the mean flow, is measured as a function of time at a fixed spatial point. To a good approximation, this can be thought of as a frozen turbulent structure advected past the probe at the mean speed U. This frozen turbulence assumption converts the signal to a representation of the velocity $u(x)$ at a fixed time. This is a good approximation for the small-scale fluctuations.

If this signal is differentiated and squared, it gives a plausible one-dimensional surrogate for the local dissipation,

$$\tilde{\varepsilon}(x) = \nu(\partial u/\partial x)^2 \tag{15}$$

This dissipation signal is observed to be highly intermittent, with bursts of intense activity alternating with inactive periods. Consider the spatial average of this local dissipation over an inertial range interval of length r,

$$\varepsilon_r = (1/r)\int_x^{x+r} \tilde{\varepsilon}(x')\,dx' \tag{16}$$

The use of a one-dimensional average in Eq. (16) is strictly for experimental convenience. A volume average would be preferable, but is difficult to measure. I return briefly to this point later.

In 1962 Obukhov[9] proposed that the 1941 Kolmogorov theory should be generalized to include the fluctuations in ε_r. In particular, he suggested that the velocity structure functions

$$C_n(r) = \langle [u(x+r) - u(x)]^n \rangle = \text{const} \cdot \langle \varepsilon_r^{n/3} \rangle \, r^{n/3} \tag{17}$$

Equation (17) can be given a plausible physical interpretation.[10] Assume that the dissipation averaged over a region of size $r \gg \eta$ exhibits the same statistics as the energy transfer due to the nonlinear terms. The dimensions of energy transfer are velocity cubed divided by length. On a scale of size r, the natural one-dimensional surrogate for the energy transfer is $[u(x+r) - u(x)]^3/r$. This suggests Eq. (17). If there is an underlying self-similarity, we expect the structure functions $C_n(r)$ to go as some power of r, but this power can be modified by the fluctuations of the spatially averaged dissipation in Eq. (17). Thus, assume that

$$C_n(r) = \text{const} \cdot r^{\xi_n} \tag{18}$$

where the exponents ξ_n remain to be determined.

For $n = 3$, the average dissipation enters, and this is constant. Thus, Eq. (17) predicts $\xi_3 = 1$. This is consistent with a firm theoretical result in the inertial range (see ref. 1, p. 140):

$$C_3(r) = -(4/5)\, \varepsilon r \tag{19}$$

For other values of n, the first factor in Eq. (17) depends on the statistical distribution of ε_r.

Since Eq. (19) is one of the few results which can be derived from the Navier–Stokes equations, it is worth outlining the essential steps in its derivation. One starts with an expression for the time derivative of the average $\langle v_i(\mathbf{x})\, v_j(\mathbf{x}+\mathbf{r}) \rangle$ in an unforced, slowly decaying flow. The Navier–Stokes equation is then used to express the time derivative, assuming homogeneity and isotropy at the small scales. This relates terms quadratic in velocities to terms cubic in velocities, which can be expressed in terms of $C_2(r)$ and $C_3(r)$ using isotropy conditions. The only remaining time derivative is the rate of decrease of kinetic energy, which is the mean dissipation rate ε. The final result is

$$C_3(r) = -(4/5)\, \varepsilon r + 6\nu \, dC_2(r)/dr \tag{20}$$

In the inertial range the viscous term can be neglected, and Eq. (19) results. Equation (20) dates back to Von Karman and Howarth in 1938, and is often called the Karman–Howarth equation. It is an important constraint on any approximate theory of turbulence.

In general we expect an infinite number of independent exponents defining the scaling of different powers of the dissipation, but it is natural to start with the second moment

$$\langle \varepsilon_r^2 \rangle = \text{const} \cdot (L/r)^{\mu} \tag{21}$$

Using the definition in Eq. (16), one sees that there is an analogy to Brownian motion with $r\varepsilon_r$ as the analogue of displacement and $\tilde{\varepsilon}(x)$ as the analogue of velocity. Using this analogy, one has that the autocorrelation function $\langle \tilde{\varepsilon}(x)\, \tilde{\varepsilon}(x+r) \rangle$ also goes as $(L/r)^{\mu}$. The exponent μ is perhaps the most direct characterization of the intermittent structure of the dissipation. If the Obukhov assumption is correct, Eqs. (17) and (19) can be combined to tell us that $\xi_6 = 2 - \mu$. This is a prediction relating two measurable exponents, and is in satisfactory agreement with experiment. The most recent experiments[11] suggest $\mu = 0.25$ with an uncertainty of about 10 %. To study other moments, however, we need to know more about the statistical properties of the dissipation.

The earliest prediction was by Kolmogorov[12] himself, who suggested that ε_r should have a log normal distribution. In particular, he predicted that

$$\xi_n = \frac{n}{3} + \frac{\mu n(3-n)}{18} \tag{22}$$

In particular, for $n = 2$, $\xi_2 = 2/3 + \mu/9$, and the energy spectrum goes as $k^{-\gamma}$, with $\gamma = 5/3 + \mu/9$.

In 1974, Kraichnan[8] discussed the physics of the 1962 Kolmogorov theory, and Mandelbrot[13] introduced a geometrical interpretation of the intermittency of the turbulent dissipation. In the simplest case the dissipation is concentrated on a homogeneous fractal of dimension $D = 3 - \mu$. In this case the correction to the 5/3 law for the spectrum is $\gamma = 5/3 + \mu/3$. This special case of fractally homogeneous turbulence was physically interpreted by Frisch *et al.*[14] and is frequently called the β-model. It has been known for at least a decade that both the log normal and β-models are oversimplifications, and that the general multifractal formalism introduced by Mandelbrot is needed to account for all of the data. This formalism has now become quite fashionable for a variety of problems, and I follow the notation of some of the more recent work.

I introduce the generalized dimensions[15] D_q. Let E_r be the total dissipation in a box of size r. I define D_q by summing this over all boxes and assuming

$$\sum_{\text{boxes}} E_r^q \sim r^{(q-1)D_q} \tag{23}$$

where r is in the inertial range. Meneveau and Sreenivasan[16] have found that the curve of D_q versus q is nearly the same for a wide variety of turbulent flows, including a wind tunnel boundary layer, the turbulent flow behind a grid, the turbulent wake behind a cylinder, and the surface layer of the atmosphere. They also found[17] that all of the data can be fit by a remarkably simple formula,

$$D_q = (1-q)^{-1} \log_2[p^q + (1-p)^q] \tag{24}$$

where p is a single free parameter. Assuming Eq. (24), the parameter p is quite accurately determined to be $p = 0.7$. The resulting D_q curve is nearly in agreement with the Kolmogorov log normal model for small q, but D_q approaches a constant for large q, as in the β-model. Their fit to the averaged data is shown in Fig. 1.

Equation (24) has a simple interpretation in terms of a two-scale Cantor set. Suppose that at each cascade step, one eddy of size r forms eight daughter eddies of size $r/2$. Suppose further that the amount of energy delivered to each daughter eddy is either p or $1 - p$ at random, and assume that this process repeats indefinitely. After n cascade steps, the sum over boxes in Eq. (23) is a sum over a binomial distribution, and gives Eq. (24).

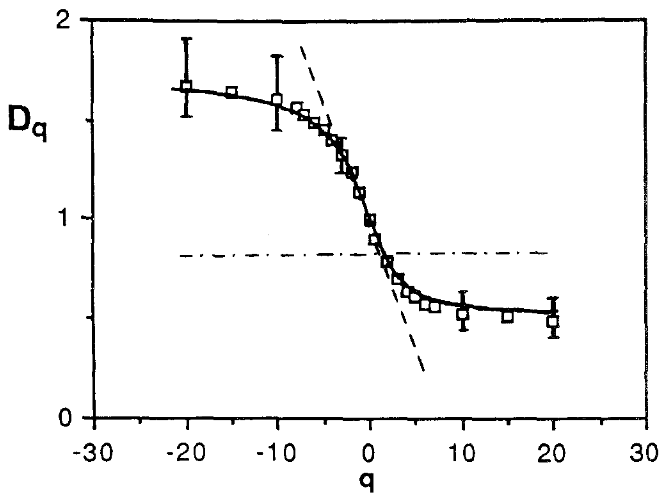

Fig. 1. The generalized dimensions for one-dimensional sections through the dissipation field in several fully developed turbulent flows. The symbols correspond to the experimental mean (which is independent of the type of flow within experimental accuracy), and the solid curve to Eq. (24) with $p = 0.7$. The dashed line corresponds to Kolmogorov's 1962 log normal model, and the horizontal dot-dashed line to the β-model, both for $\mu = 0.25$. This is Fig. 1 of ref. 17.

The multifractal formalism can be carried further using the now familiar $f(\alpha)$ curve,[18] but this is not useful here. I prefer to mention briefly the limitations of the formalism when applied to real data, and finally to suggest the physical challenge of the results. One important limitation is the restriction to one component of the velocity derivative tensor as a surrogate for the dissipation. Siggia[19] has shown that, for isotropic turbulence, there are four invariants which can be formed of fourth order in the velocity derivative tensor $\partial v_i/\partial x_j$. These invariants can be expressed in terms of the vorticity and rate of strain, and given some physical interpretation. At low Reynolds numbers, Kerr[20] has found that different invariants scale quite differently with Reynolds number. At high enough Reynolds numbers, it seems reasonable that all components should scale in the same way, since the most rapidly growing invariant should dominate. It would not be surprising, however, if the necessary Reynolds numbers were very high, even by geophysical standards. A second limitation is the use of a one-dimensional cut through the multifractal structure. Recently Prasad *et al.*[21] have made a two-dimensional cut through the dissipation of a passive scalar, and have studied two different components of the scalar derivative. They found that the two-dimensional cut reduced the fluctuations in the observed D_q, but that the averaged results from a one-dimensional cut with a single component were not essentially changed. Finally, the corrections to the frozen turbulence assumption are not known, and we do not know how accurate is the original Obukhov assumption. With all of these qualifications, I am inclined to accept the D_q curve of Eq. (24) as a reasonable approximation to a universal multifractal structure for the small scales of turbulence.

Finally, what is the physics? Independent of the simple model, the D_q curve tends to a constant value of about 0.5 rather quickly as q increases. This indicates that the ratio of the dissipation on scale $r/2$ to the dissipation on scale r has a robust upper bound. This of course depends on the reliability of measurements of high moments of the dissipation. If true, however, it has strong implications for the dynamics of forming small scales in turbulence. With the simple model of Eq. (24), this upper bound is the only free parameter. The two-scale Cantor set is undoubtedly an oversimplification, as is the assumption of a factor of 2 change in scale size. Despite all these caveats, I suspect that the essential features of this simple model are telling us something quite basic about the dynamics of turbulence. We are, a long way from understanding this type of question directly from the Navier–Stokes equations. My own temptation is to introduce dynamical models at an intermediate level of complexity, and to see if any of these models can exhibit solutions in agreement with the observed structure of the small scales in high-Reynolds-number turbulence.

To summarize, I have suggested that fully developed turbulence is contained within the Navier–Stokes equations. I have reviewed the strong phenomenological evidence for the 1941 Kolmogorov theory. I have briefly examined direct numerical simulations, with an emphasis on possible mechanisms by which small scales are created, and have suggested that ribbons of vorticity are formed. I then returned to the experiments on the multifractal structure of the dissipation at high Reynolds number, and suggested that the phenomenological evidence for some kind of dynamical self-similarity is quite strong. I emphasized, however, that we have made no theoretical progress in obtaining such self-similarity from the underlying Navier–Stokes equations. Without the help of experiment, we would have little hint that it is there.

After finishing this paper, I became aware of another recent review[22] which partially overlaps this one. The subjects covered, and the points of view expressed, are somewhat different, but represent an overall view of fully developed turbulence which is broadly similar. Both papers completely ignore one important aspect of turbulence theory, namely the statistical theories which have as their goal to obtain and solve equations for statistical quantities such as the energy spectrum. For a discussion of the fundamental theoretical problems involved in these theories, see the 1977 paper by Kraichnan.[23] For recent interesting applications, see the work of Dannevik *et al.*[24] The spirit of these applications is to assume self-similarity and the 5/3 law of the 1941 Kolmogorov theory, and then to calculate the scaling function of Eq. (14) as well as other low-order statistical averages over turbulent fluctuations. This is a very different point of view than taken in the present paper.

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REFERENCES

1. L. D. Landau and L. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, 1987).
2. M. Van Dyke, *An Album of Fluid Motion* (Parabolic Press, Stanford, California, 1982).

3. J. Koplik, J. R. Banavar, and J. F. Willemsen, *Phys. Rev. Lett.* **60**:1282 (1988).
4. F. H. Champagne, *J. Fluid Mech.* **86**:67 (1978).
5. A. Pumir and E. D. Siggia, *Phys. Fluids* **30**:1606 (1987).
6. A. Pumir and R. M. Kerr, *Phys. Rev. Lett.* **58**:1636 (1987).
7. R. M. Kerr, *Phys. Rev. Lett.* **59**:783 (1987).
8. R. H. Kraichnan, *J. Fluid Mech.* **62**:305 (1974).
9. A. M. Obukhov, *J. Fluid Mech.* **13**:77 (1962).
10. M. Nelkin and T. L. Bell, *Phys. Rev. A* **17**:363 (1978).
11. F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, *J. Fluid Mech.* **140**:63 (1984).
12. A. N. Kolmogorov, *J. Fluid Mech.* **13**:82 (1962).
13. B. Mandelbrot, *J. Fluid Mech.* **62**:331 (1974).
14. U. Frisch, P. L. Sulem, and M. Nelkin, *J. Fluid Mech.* **87**:719 (1978).
15. H. G. E. Hentschel and I. Procaccia, *Physica* **8D**:435 (1983).
16. C. Meneveau and K. R. Sreenivasan, *Nucl. Phys. B (Proc. Suppl.)* **2**:49 (1987).
17. C. Meneveau and K. R. Sreenivasan, *Phys. Rev. Lett.* **59**:1424 (1987).
18. T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, *Phys. Rev. A* **33**:1141 (1986).
19. E. D. Siggia, *Phys. Fluids* **24**:1934 (1981).
20. R. M. Kerr, *J. Fluid Mech.* **153**:31 (1985).
21. R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, *Phys. Rev. Lett.* **61**:74 (1988).
22. U. Frisch, *Phys. Scripta* **T9**:137 (1985) [in French; English translation] in *Dynamical Systems, A Renewal of Mechanism*, S. Diner, G. Fargue, and G. Lochak, eds. (World Publishing, Singapore, 1986), p. 13.
23. R. H. Kraichnan, *J. Fluid Mech.* **83**:349 (1977).
24. W. P. Dannevik, V. Yakhot, and S. A. Orszag, *Phys. Fluids* **30**:2021 (1987).